

# A cloned qutrit and its utility in information processing tasks

Sovik Roy <sup>1,4\*</sup>, Nirman Ganguly <sup>3,4†</sup>, Atul Kumar <sup>2‡</sup>, Satyabrata Adhikari <sup>2§</sup>, A. S. Majumdar <sup>4¶</sup>

<sup>1</sup> Techno India, Salt Lake City, Kolkata - 91, India

<sup>2</sup> Indian Institute of Technology, Rajasthan, Jodhpur- 11, India

<sup>3</sup> Heritage Institute of Technology, Kolkata - 107, India

<sup>4</sup> S. N. Bose National Centre for Basic Sciences, Salt Lake, Kolkata-98, India

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## Abstract

We analyze the efficacy of an output as a resource from a universal quantum cloning machine in information processing tasks such as teleportation and dense coding. For this, we have considered the  $3 \otimes 3$  dimensional systems. The output states are found to be NPT states for a certain range of machine parameters. Using the output state as an entangled resource, we also study the optimal fidelities of teleportation and capacities of dense coding protocols with respect to the machine parameters and make some interesting observations. Our work is motivated from the fact that the cloning output can be used as a resource in quantum information processing and adds a valuable dimension to the applications of cloning machines.

## 1 Introduction

Quantum entanglement apart from being central to the investigations of the fundamentals of quantum mechanics, has also been used as a resource enabling efficient quantum information

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\*sovikr@rediffmail.com

†nirmanganguly@gmail.com

‡atulk@iitj.ac.in

§satya@iitj.ac.in

¶archan@bose.res.in

processing protocols like quantum teleportation [1], superdense coding [2], cryptography [3], quantum cloning [4] etc. The question of usefulness of states for quantum teleportation [5], dense coding [6] and quantum cryptography [7] have also been studied in detail. Although most of the information processing protocols put emphasis on the use of two level systems as the fundamental units of quantum information transfer, there is a growing interest among the quantum information community to study multi-level or higher dimensional systems for quantum information processing [8]. This attributes to the fact that in the higher dimensions quantum information processes are supposed to be more efficient in certain situations. For example, a quantum state in a large dimensional space contains more information than one in small dimensional space [9]. Moreover, the present experimental context also makes it reasonable to consider the manipulation of more-than-two-level quantum information carriers.

In this article, we address the problem of using higher dimensional systems for quantum information processing. Central to our investigation is the usefulness of mixed states of two qutrits, obtained as an output from Buzek - Hillery universal quantum cloning machine [10], as resources for quantum teleportation and dense coding protocols. Our analysis of teleportation and dense coding using higher dimensional systems provides some interesting aspects about the outputs obtained from the cloning machine when used as resources in quantum information processing. We show that the non-optimal output state obtained from the Buzek - Hillery cloning machine can be used more efficiently as a resource for quantum information processing in comparison to the optimal output state obtained from the Buzek - Hillery cloning machine. Surprisingly, our results show that the optimal teleportation fidelity for a distilled non-optimal output state as a resource is more than the optimal teleportation fidelity for a distilled optimal output state as a resource for a certain range of machine parameters. For dense coding one can successfully use only a distilled non-optimal output state as a resource and not the distilled optimal output state. We believe that the results obtained in this article would add another important dimension to the applicability of quantum cloning machines to quantum information processing [11].

In section 2, we give a brief review of Buzek Hillery universal quantum cloning machine and discuss the entanglement properties of the output states obtained from the cloning machine. Section 3 and Section 4, deal with the analysis of the optimal and non-optimal output state, respectively for quantum information processing. We conclude the article with results and discussions in Section 5.

## 2 Entanglement properties of the two qutrit state obtained from the Buzek - Hillery universal quantum cloning machine (BH-UQCM):

In order to facilitate the discussion of our analysis and results, we briefly describe V. Buzek and M. Hillery's universal quantum cloning machine (UQCM) for arbitrary higher dimensional Hilbert space[10]. If we consider the initial state of the cloning machine as  $|X\rangle_x$ , then the cloning transformation is given by

$$|\psi_i\rangle_a |0\rangle_b |X\rangle_x \longrightarrow \lambda |\psi_i\rangle_a |\psi_i\rangle_b |X_i\rangle_x + \mu \sum_{j \neq i}^2 \{ |\psi_i\rangle_a |\psi_j\rangle_b + |\psi_j\rangle_a |\psi_i\rangle_b \} |X_j\rangle_x \quad (1)$$

where  $|\psi_i\rangle_a$  is the initial state to be cloned and  $|0\rangle_b$  is the blank copy on which the input state has to be cloned with real parameters  $\lambda$  and  $\mu$ . The unitarity of the transformation ensures the following relation between the parameters  $\lambda$  and  $\mu$ ;

$$\lambda^2 + 4\mu^2 = 1 \quad (2)$$

The values of the parameters  $\lambda$  and  $\mu$  which guarantee the universality and optimality of the transformation is given by  $\lambda^2 = \frac{1}{2}$  and  $\mu^2 = \frac{1}{8}$ [10].

We now proceed to discuss the entanglement properties of a two qutrit system generated as the output of the UQCM (1). For this, we consider a single qutrit system as a input to the cloning machine, namely

$$|\varphi\rangle = \frac{1}{\sqrt{3}} [|0\rangle + |1\rangle + |2\rangle] \quad (3)$$

The output corresponding to two qutrit system depends on the machine parameters and can be given by

$$\begin{aligned} \rho_{ab}^{out} = & \frac{(1-4\mu^2)}{3} \sum_{i=0}^2 |i, i\rangle \langle i, i| + \frac{2}{3} \mu^2 \sum_{i \neq j}^2 |i, j\rangle [\langle i, j| + \langle j, i|] \\ & + \frac{\sqrt{(1-4\mu^2)}}{3} \mu \left[ \sum_{i \neq j}^2 |i, i\rangle (\langle i, j| + \langle j, i|) + \sum_{i \neq j}^2 |i, j\rangle (\langle i, i| + \langle j, j|) \right] \\ & + \frac{\mu^2}{3} \left[ \sum_{i \neq j}^2 |i, j\rangle (\langle i, j+1| + \langle j+1, i| + \langle i+1, j+1| + \langle j+1, i+1|)(\text{mod}2) \right] \end{aligned} \quad (4)$$

Using the positive partial transposition criteria [12] with respect to the system  $a$ , we show that at least one of the two obtained eigenvalues

$$E_1 = \frac{1+4\mu^2}{6} - \frac{1}{6}\sqrt{1+24\mu^2-104\mu^4+32\sqrt{-(2\mu-1)(2\mu+1)}\mu^3} \quad (5)$$

and

$$E_2 = \frac{1-5\mu^2}{6} - \frac{1}{6}\sqrt{1-6\mu^2+25\mu^4-16\sqrt{-(2\mu-1)(2\mu+1)}\mu^3} \quad (6)$$

are always negative when  $\mu \in (0, \frac{1}{2}]$ . For example, the eigenvalue  $E_1$  is negative when  $\mu \in (0, \frac{6+\sqrt{2}}{17})$  and the eigen value  $E_2$  is negative when either  $\mu \in (0, \frac{1}{2\sqrt{2}})$  or  $\mu \in (\frac{1}{2\sqrt{2}}, \frac{1}{2}]$ . Therefore, the state (4) is a NPT state for  $\mu \in (0, \frac{1}{2}]$ . The fact that the cloned two-qutrit system generated from the UQCM is a NPT state motivates us to analyse its utility for quantum information processing tasks such as teleportation and dense coding.

### 3 Analysis of quantum teleportation using the optimal output state of BH-UQCM:

The output two-qutrit state obtained from the cloning machine will be an optimal state  $\rho_{ab}^{opt}$  for  $\mu^2 = \frac{1}{8}$ . From (4), it is easy to show that such an output state would be an NPT state according to [12].

Moreover, the fully entangled fraction for any arbitrary state  $\rho$  is defined as

$$F(\rho) = \text{Max}_{\phi} \langle \phi | \rho | \phi \rangle \quad (7)$$

where maximum is taken over all the maximally entangled basis states  $\phi$ . Further, we know that a qutrit system can be used as a teleportation channel, if its fully entangled fraction i.e.  $F(\rho)$  is greater than  $\frac{1}{3}$  [13]. In order to confirm the utility of the two qutrit optimal output state  $\rho_{ab}^{out}$  in teleportation, we now proceed to calculate its fully entangled fraction.

The maximally entangled orthonormal basis for two-qutrit system can be represented as [14]

$$|\phi_{xy}\rangle = \frac{1}{\sqrt{3}} \sum_{j=0}^2 \xi^{jy} |j, j+x\rangle, \quad x, y = 0, 1, 2 \quad (8)$$

where,  $\xi := e^{\frac{2\pi i}{3}}$  and  $\{|0\rangle, |1\rangle, |2\rangle\}$  is an orthonormal basis for the space of one qutrit. Therefore, considering the optimal output state  $\rho_{ab}^{opt}$  and using (7), (8) and

$$\langle \phi_{ij} | \rho | \phi_{ij} \rangle = \text{Tr}(\rho | \phi_{ij} \rangle \langle \phi_{ij} |) \quad (9)$$

we find that,  $F(\rho_{ab}^{opt}) = \frac{1}{6} < \frac{1}{3}$ . Hence, the state  $\rho_{ab}^{opt}$  cannot be used as a resource for quantum teleportation.

### 3.1 Analysis of dense coding using the optimal output state of BH-UQCM:

Quantum dense coding is the simplest example where one can utilize a shared entangled system to efficiently transfer the information between the users. For any  $3 \otimes 3$  dimensional system, the capacity of dense coding for any given shared state  $\rho_{ab}$  can be defined as,

$$\chi = \log_2 3 + S(\rho_b) - S(\rho_{ab}) \quad (10)$$

where  $S(\rho_b)$  is the von - Neumann entropy of the reduced system  $\rho_b$  and  $S(\rho_{ab})$  is von - Neumann entropy for the joint state  $\rho_{ab}$ . A shared quantum state is thus said to be useful for dense coding, if the corresponding capacity  $\chi$  is more than  $\log_2(3)$ . From (10), it is clear that such states are precisely those for which  $S(\rho_b) - S(\rho_{ab}) > 0$ .

For the state  $\rho_{ab}^{opt}$ , we find that  $S(\rho_b^{opt}) - S(\rho_{ab}^{opt}) = -0.43872 < 0$  and hence the state  $\rho_{ab}^{opt}$  is not useful for the dense coding protocol.

### 3.2 Construction of a filter and analysis of the optimal filtered state in teleportation and dense coding:

In Section 3, we have seen that the optimal output state  $\rho_{ab}^{opt}$  is not useful for teleportation and dense coding protocols. For  $2 \otimes 2$  dimensional systems, any state can be made useful for teleportation [15]. However, for qutrit systems and other higher dimensional systems such possibilities have not been explored.

In this section, we try to distill the optimal output state using an appropriate filter to check the suitability of the filtered state for the teleportation and dense coding protocols. Horodecki *et al.* showed that any state violating the reduction criteria is distillable [16]. Using the above criteria, it is easy to find that the state  $\rho_{ab}^{opt}$  violates the reduction criteria and hence is also distillable.

The distilled  $\rho_{ab}^{opt}$  can be obtained by calculating the eigenvector corresponding to the suitable negative eigenvalue of the state  $(\rho_a^{opt} \otimes I - \rho_{ab}^{opt})$  and subjecting the state to the appropriate filter. Further, we know that if  $\rho$  is any state and  $A$  is a filter then a new state  $\rho'$  can be found

as

$$\rho' = \frac{A^\dagger \otimes I \rho A \otimes I}{\text{Tr}(\rho A A^\dagger \otimes I)} \quad (11)$$

where  $A$  is the filter [16]. Following the procedures described in [16], we construct the filter for the optimal state  $\rho_{ab}^{opt}$ . The filter is denoted by  $A_{opt}$  and is given by

$$A_{opt} = \begin{pmatrix} \sqrt{3}(\frac{3}{2} - \frac{\sqrt{29}}{2}) & \sqrt{3}(-\frac{7}{2} + \frac{\sqrt{29}}{2}) & -\sqrt{3} \\ \sqrt{3}(\frac{7}{2} - \frac{\sqrt{29}}{2}) & \sqrt{3}(-\frac{3}{2} + \frac{\sqrt{29}}{2}) & \sqrt{3} \\ \sqrt{3}(\frac{5}{2} - \frac{\sqrt{29}}{2}) & \sqrt{3}(-\frac{5}{2} + \frac{\sqrt{29}}{2}) & 0 \end{pmatrix} \quad (12)$$

Using (11), we find the filtered form of  $\rho_{ab}^{opt}$  denoted by  $(\rho')_{ab}^{opt}$  such that

$$(\rho')_{ab}^{opt} = \frac{A_{opt}^\dagger \otimes I \rho_{ab}^{optout} A_{opt} \otimes I}{\text{Tr}(\rho_{ab}^{optout} A_{opt} A_{opt}^\dagger \otimes I)} \quad (13)$$

Interestingly, the filtered optimal state (13) is found to be suitable for teleportation since the fully entangled fraction of  $(\rho')_{ab}^{opt} = 0.38789 > \frac{1}{3}$ . Moreover, the optimal teleportation fidelity for any arbitrary state  $\rho$  in 3 dimensional system is defined as

$$f(\rho) = \frac{3F(\rho) + 1}{4} \quad (14)$$

Therefore, in our case the optimal teleportation fidelity of the state  $(\rho')_{ab}^{opt}$  is given by

$$f((\rho')_{ab}^{opt}) = \frac{3F((\rho')_{ab}^{opt}) + 1}{4} = 0.5409 \quad (15)$$

Similarly for the dense coding protocol using (10) and (13), we see that  $S((\rho')_{ab}^{opt}) - S((\rho')_{ab}^{opt}) = -0.3327 < 0$ . Therefore, the filtered optimal state is still not useful for dense coding.

## 4 Non - optimal output state as a resource in teleportation and dense coding:

The output state generated from the cloning machine depends on the machine parameters  $\lambda$  and  $\mu$ . For all the values of the parameter  $\mu$  (except for  $\mu^2 = \frac{1}{8}$ ) in the range  $\mu \in (0, \frac{1}{2}]$ , the output state  $\rho_{ab}^{out}$  is a non-optimal state. Although we have shown that the state  $\rho_{ab}^{out}$  is always a NPT state in this range, i.e. when  $\mu \in (0, \frac{1}{2}]$ , the fully entangled fraction (7) of the state  $\rho_{ab}^{out}$  is  $F(\rho_{ab}^{out}) = \frac{4\mu^2}{3}$  which never exceeds  $\frac{1}{3}$ . Hence, the non-optimal state  $\rho_{ab}^{out}$  is also not useful to be used as a resource for teleportation process.

In order to see whether the state  $\rho_{ab}^{out}$  is useful in dense coding or not, we plot a graph between  $S(\rho_b^{out}) - S(\rho_{ab}^{out})$  and the machine parameter  $\mu$ , where  $\rho_b^{out}$  is the reduced density operator of the two qutrit output state  $\rho_{out}^{ab}$  and is given by

$$\rho_b^{out} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3}\mu(2\sqrt{1-4\mu^2} + \mu) & \frac{1}{3}\mu(2\sqrt{1-4\mu^2} + \mu) \\ \frac{1}{3}\mu(2\sqrt{1-4\mu^2} + \mu) & \frac{1}{3} & \frac{1}{3}\mu(2\sqrt{1-4\mu^2} + \mu) \\ \frac{1}{3}\mu(2\sqrt{1-4\mu^2} + \mu) & \frac{1}{3}\mu(2\sqrt{1-4\mu^2} + \mu) & \frac{1}{3} \end{pmatrix} \quad (16)$$

It is clear from the fig. (1) that the two qutrit non optimal state  $\rho_{ab}^{out}$  is not useful for dense coding when  $\mu \in (0, \frac{1}{2}]$ .

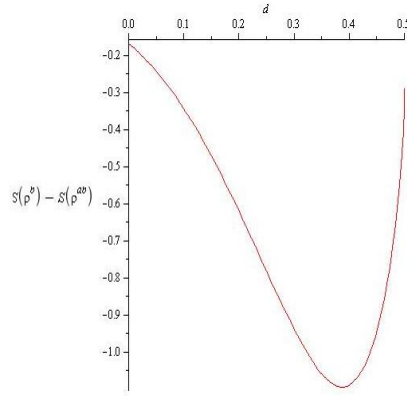


Figure 1: The figure shows that the difference  $S(\rho_b) - S(\rho_{ab})$  always lies in the fourth quadrant of the cartesian plane with respect to the values of  $\mu$  in the range  $(0, \frac{1}{2}]$ .

#### 4.1 Construction of a filter and analysis of the non - optimal filtered states in teleportation and dense coding:

As the non optimal output state also cannot be used for information transfer, we try to find out whether it is possible to distill the non optimal output state using an appropriate filter so that the distilled system can be used for quantum information protocols.

Similar to the optimal state output, the non optimal state (4) also violates the reduction criteria [16]. Therefore, it is possible to distill the non optimal output state  $\rho_{ab}^{out}$  of two qutrits. For this, we calculate the eigenvector corresponding to the suitable negative eigenvalue of the state  $(\rho_a^{out} \otimes I - \rho_{ab}^{out})$  and by subjecting the state to an appropriate filter. In this case, the obtained eigenvalue is expressed as

$$E = \frac{1-3\mu^2}{6} + \frac{1}{3}\sqrt{-(2\mu-1)(2\mu+1)d} - \frac{1}{6}\sqrt{1-18\mu^2+4\sqrt{-(2\mu-1)(2\mu+1)}\mu+113\mu^4-44\mu^3\sqrt{-(2\mu-1)(2\mu+1)}} \quad (17)$$

which is always negative for  $\mu \in (\frac{6+\sqrt{2}}{17}, \frac{1}{2}]$ . Hence, following [16], we construct the filter  $A_{nonopt}$  to distill the state  $\rho_{ab}^{out}$  where

$$A_{nonopt} = \sqrt{3} \begin{pmatrix} 1 & -k & -k \\ -k & 1 & -k \\ -k & -k & 1 \end{pmatrix} \quad (18)$$

and  $k = \frac{11\mu^2-2\sqrt{1-4\mu^2}\mu+\sqrt{1-18\mu^2+4\sqrt{1-4\mu^2}\mu+113\mu^4-44\mu^3\sqrt{1-4\mu^2}-1}}{4\mu^2}$ ;  $\mu \in (\frac{6+\sqrt{2}}{17}, \frac{1}{2}]$ . This filter thus transforms  $\rho_{ab}^{out}$  to its filtered form  $\tau_{ab}^{out}$  given by

$$\tau_{ab}^{out} = \frac{A_{nonopt}^\dagger \otimes I \rho_{ab}^{out} A_{nonopt} \otimes I}{Tr(\rho_{ab}^{out} A_{nonopt} A_{nonopt}^\dagger \otimes I)} \quad (19)$$

Therefore, fully entangled fraction of the distilled state  $\tau_{ab}^{out}$  is

$$F(\tau_{ab}^{out}) = \frac{4[\mu^2(2(1-t_1)+\mu^2(22t_1-31))+t_2(10-110\mu^2-6t_1)+198\mu^4]}{3[(1-k)+t_2(6-68\mu^2+94\mu^4-4t_1+12t_1\mu^2)+\mu^2(6t_1-9-5\mu^2+23t_1\mu^2)]} \quad (20)$$

where  $t_2 = \sqrt{1-4\mu^2}.\mu$  and  $t_1 = \sqrt{1-18\mu^2+4t_2+113\mu^4-44\mu^2.t_2}$ . Evidently, for  $\mu \in (\frac{6+\sqrt{2}}{17}, \frac{1}{2}]$ , the states  $\tau_{out}^{ab}$  are always suitable for teleportation since  $F(\tau_{out}^{ab}) > \frac{1}{3}$ . Moreover, the optimal teleportation fidelity of  $\tau_{out}^{ab}$  is given by

$$f(\tau_{out}^{ab}) = \frac{3F(\tau_{out}^{ab})+1}{4} \quad (21)$$



The plot between the optimal teleportation fidelity and  $\mu$  in fig. (2) also confirms the utility of  $\tau_{ab}^{out}$  as an entangled resource for teleportation protocol.

In order to check the usefulness of the state  $\tau_b^{out}$  for dense coding, we show that  $S(\tau_b^{out}) - S(\tau_{ab}^{out}) > 0$ , for  $\mu \in (\frac{6+\sqrt{2}}{17}, \frac{1}{2}]$ . Hence, the state  $\tau_{ab}^{out}$  can also be used as a resource in dense coding protocol as also evident from the fig. (3). Interestingly, the optimal fidelity obtained

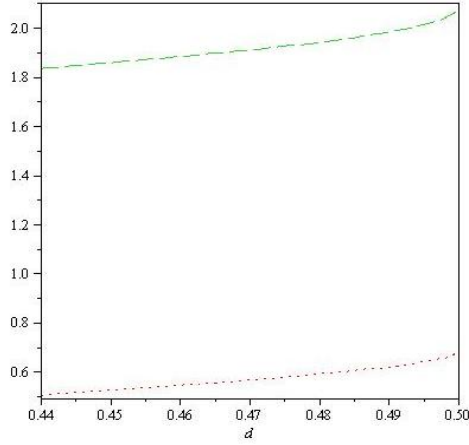


Figure 2: In the figure the dotted lines represent the capacity of dense coding i.e.  $\chi(\tau_{ab}^{out})$  of the non optimal states and dashed line represent the teleportation fidelities i.e.  $f(\tau_{ab}^{out})$  of the said state with respect to  $\mu$ .

in case of teleportation using the distilled non-optimal output state as an entangled resource is greater than the optimal fidelity obtained in case of teleportation using the distilled optimal output state. Moreover, the distilled optimal output state cannot be used for dense coding but the distilled non-optimal output state can be successfully used as a resource for dense coding.

## 5 Summary and Discussion:

We have studied the entanglement properties and the usefulness of a two qutrit output state generated through Buzek - Hillery quantum cloning machine for quantum information processing. Although we found that the optimal as well as non-optimal output states are not useful for information processing protocols such as teleportation and dense coding, the distilled output states using appropriate filters can be used as an entangled resource for information processing

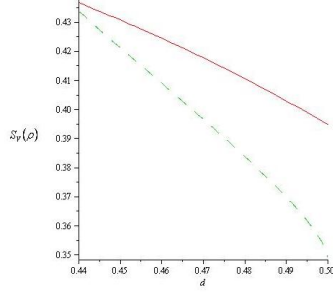


Figure 3: The ordinate of the figure represents the von - Neumann entropy of a state  $\rho$ , where solid line corresponds when  $\rho = \rho_{ab}^{out}$  and dotted line corresponds when  $\rho = \tau_{ab}^{out}$ .

for certain range of machine parameters. It is interesting to note that while the distilled optimal output state can only be used for teleportation, the distilled non-optimal output state can be used for teleportation as well as dense coding. Surprisingly, the optimal teleportation fidelity obtained using the distilled non-optimal output state exceeds the optimal teleportation fidelity obtained using the distilled optimal output state for certain values of machine parameter  $\mu$ .

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